# Khovanskii-finite rational curves of arithmetic genus 2

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- Applications: Mirror symmetry, numerical algebraic geometry, Seshadri constants, ...

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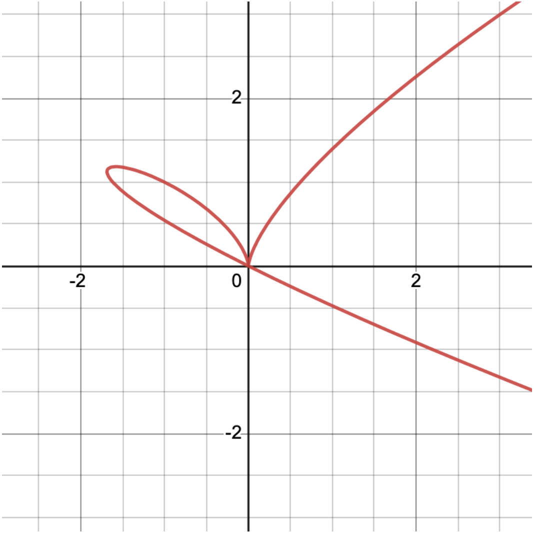

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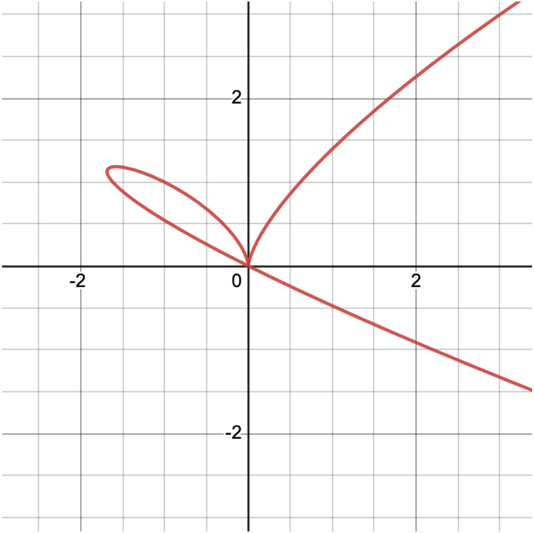

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- Rational curves:
  - Rational curves of arithmetic genus 0 and 1: Yes [Ilten, Wrobel]
  - Very general rational quartic plane curve : No [Ilten, Wrobel]
  - **Our work:** Rational curves of arithmetic genus 2

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  - $\nu(ab) = \nu(a) + \nu(b)$,
  - $\nu(a+b) \geq \min\{\nu(a), \nu(b)\}$,
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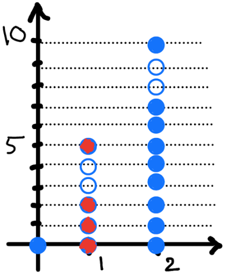

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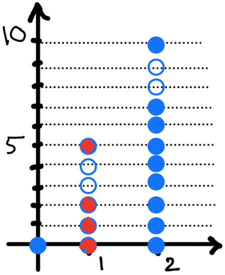


- $\nu$ is **Khovanskii-finite** if the value semigroup is full-rank and finitely-generated.

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$$\begin{aligned} \nu_{(\alpha:\beta)} : R(L)\backslash\{0\} &\to \mathbb{Z}\times\mathbb{Z}, \\ f \in L^k &\mapsto (k, \operatorname{ord}_{(\alpha:\beta)} f). \end{aligned}$$

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- [Ilten, Wrobel] K-finite criterion: Given $L$ and $(\alpha : \beta)$

  $\nu_{(\alpha:\beta)}$ on $R(L)$ is K-finite $\iff$ there is $k$ with $(\beta x - \alpha y)^{dk} \in L^k$

# Main result

- **Theorem [Ilten, M.]:**
  $X \subset \mathbb{P}^n$: non-degenerate rational curve of degree $d$ and arithmetic genus 2 over a number field with degree $\ell$.
  Then, in the K-finite criterion of Ilten-Wrobel we need only consider

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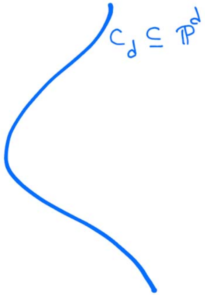
- Proof: reduce to degree $d$ curves in $\mathbb{P}^{d-2}$

$$\begin{array}{c} X' \subset \mathbb{P}^{d-2} \\ \wr\big\downarrow \\ X \subset \mathbb{P}^n \end{array}$$

# Parametrizing degree $d$ curves in $\mathbb{P}^{d-2}$



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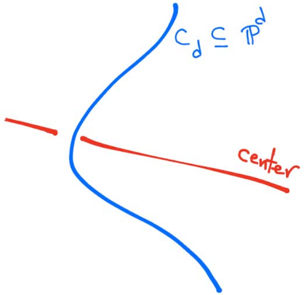

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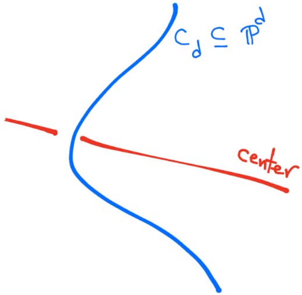

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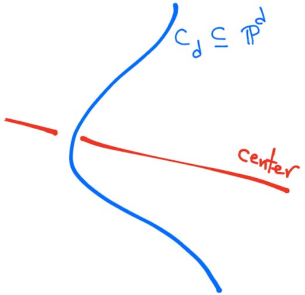


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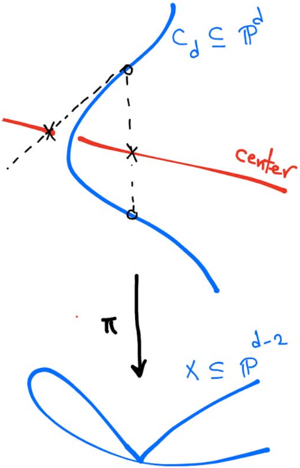

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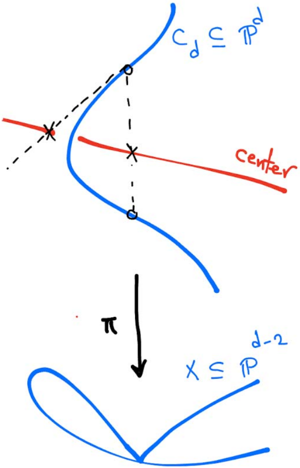


$$(x:y) \mapsto (x^d - ax^{d-1}y - y^d : x^{d-2}y^2 : \cdots : xy^{d-1}) \in \mathbb{P}^{d-2}$$

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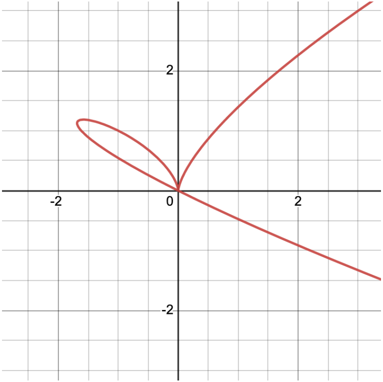


Degree: $d = 5$, Curve: $a = 5$, Valuation: $\nu_{(1:1)}$