Connectedness of Fano schemes of matrices of bounded rank

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November 2024

Motivation - Classification of matrices

$\mathbb{K}$: any algebraically closed field

- Q: $n \times n$ matrix over $\mathbb{K}$

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$$\text{matrix} \quad \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix} \rightsquigarrow \begin{pmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{pmatrix} \quad \begin{array}{l} \text{Jordan} \\ \text{form} \\ \lambda_1, \lambda_2 \in \mathbb{K} \end{array}$$

Motivation (cont'd)

- Q_1, Q_2: $n \times n$ matrices over $\mathbb{K}$

$$\begin{tikzcd} \mathbb{K}^n \arrow[r, bend left, "Q_1"] \arrow[r, bend right, "Q_2"'] & \mathbb{K}^n \end{tikzcd}$$

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$$\begin{tikzcd} \mathbb{K}^n \arrow[r, bend left, "Q_1"] \arrow[r, bend right, "Q_2"'] & \mathbb{K}^m \end{tikzcd}$$

– Classify (Q_1, Q_2) up to (AQ_1B^{-1}, AQ_2B^{-1}), $(A, B) \in \mathrm{GL}(m) \times \mathrm{GL}(n)$
– Kronecker-Weierstrass Canonical form

Linear spaces of matrices

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- Here V is one of
  - $M_{m,n}$: space of $m \times n$ rectangular matrices over $\mathbb{K}$
  - $\mathcal{S}_n$: space of $n \times n$ symmetric matrices ($B = A$, $\mathrm{char}\mathbb{K} \neq 2$)
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- Classification methods
  - Canonical form
  - **Moduli approach**

Example (rectangular matrices)

- $V = M_{m,n} \quad m = n = r = 5$

$$\begin{pmatrix} * & * & * & * & 0 \\ * & * & * & * & 0 \\ * & * & * & * & 0 \\ * & * & * & * & 0 \\ * & * & * & * & 0 \end{pmatrix}$$

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0-compression space   1-compression space

- For $0 \leqslant s \leqslant r-1$, an s-compression space is

$$\begin{array}{c c} & \begin{array}{cc} r-s-1 & s+1+n-r \end{array} \\ \begin{array}{c} s \\ m-s \end{array} & \left(\begin{array}{c|c} * & * \\ \hline * & 0 \end{array}\right) \end{array}$$

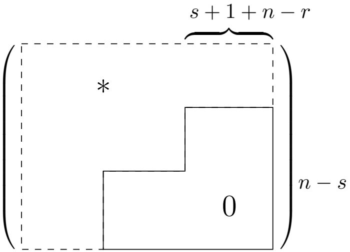
Example (symmetric/alternating case)

- $V = \mathcal{S}_n$ or $\mathcal{A}_n$ $\quad n = r = 5$

$$\begin{pmatrix} * & * & * & * & 0 \\ * & * & * & * & 0 \\ * & * & * & * & 0 \\ * & * & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} * & * & * & * & * \\ * & * & * & 0 & 0 \\ * & * & * & 0 & 0 \\ * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \end{pmatrix}$$

0-compression space $\qquad$ 1-compression space

- For $0 \leqslant s \leqslant \frac{r-1}{2}$, a sym/alt s-compression space is



Example of a non-compression space

$$\begin{pmatrix} 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & -a & 0 & c \\ 0 & 0 & 0 & -b & -c & 0 \\ 0 & -a & -b & 0 & 0 & 0 \\ a & 0 & -c & 0 & 0 & 0 \\ b & c & 0 & 0 & 0 & 0 \end{pmatrix}_{6\times 6}$$

a, b, c range over $\mathbb{K}$

Previous results on linear spaces of matrices

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- Ilten & Chan (2015) : Fano schemes of rectangular matrices of bounded rank

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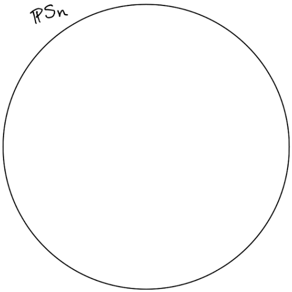
$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{12} & x_{22} & & x_{2n} \\ \vdots & & \ddots & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{nn} \end{pmatrix}$$



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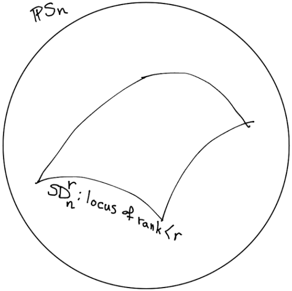

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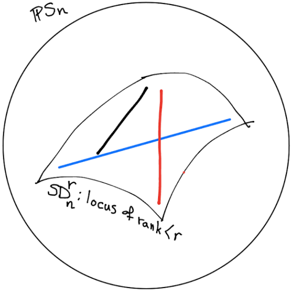

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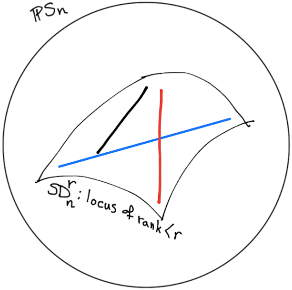

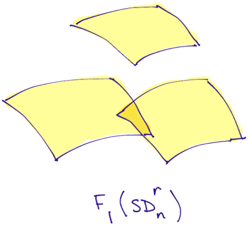

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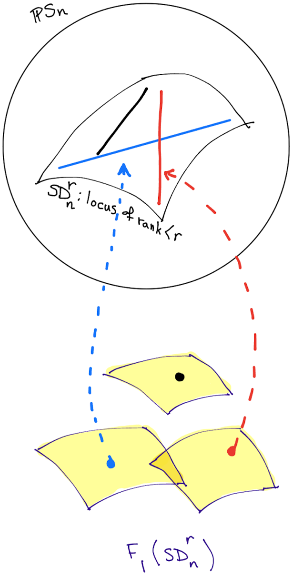

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- Provide a unified framework for proving previous results (geometric proofs)
- Improve previous results

Connectedness - Theorem (M.)

- Fix m, n, r, k
- Let $X =$

  vanishing of $r \times r$ minors of the generic $m \times n$ matrix
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There is a graph $\mathcal{G}$ associated to the scheme $\mathbf{F}_k(X)$ such that

$$\text{connected components of } \mathbf{F}_k(X) \quad \overset{1-1}{\longleftrightarrow} \quad \text{connected components of } \mathcal{G}$$

Connectedness - Example

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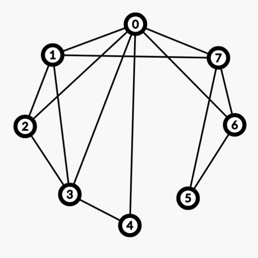
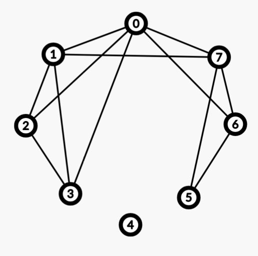
- Example: $\mathbf{F}_k(\mathrm{SD}^{15}_{15})$



$k = 70$



$k = 71$

Thank you!

Example: $\mathbf{F}_1(\mathrm{SD}_3^3)$

- ```
  R=QQ[x_1..x_6];
  Fano(1,ideal(det(genericSymmetricMatrix(R,x_1,3))))
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- Output: 70 generators in 15 variables

$x_9x_{12} - x_8x_{13} + x_5x_{14}$

$x_9x_{11} - x_7x_{13} + x_4x_{14}$

$x_8x_{11} - x_7x_{12} + x_2x_{14}$

$\vdots$

$3x_1x_2x_4 - x_0x_2x_5 - 2x_0x_2x_7-$

$\quad 2x_0x_6x_7 + 2x_0x_3x_11 - x_0^2x_13$

$3x_1^2x_4 - 2x_0x_1x_5 - x_0x_6^2 - 4x_0x_1x_7+$

$\quad 2x_0^2x_8 + x_0x_3x_10 + x_0^2x_11$

$x_1^2x_3 - 2x_0x_1x_6 + x_0^2x_10$
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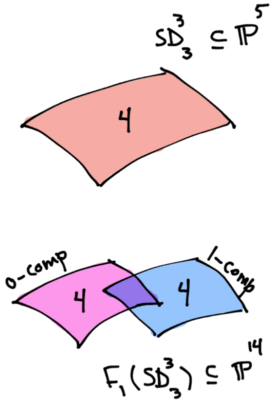
$$3x_1^2x_4 - 2x_0x_1x_5 - x_0x_6^2 - 4x_0x_1x_7 + \\ 2x_0^2x_8 + x_0x_3x_1 0 + x_0^2x_1 1$$
$$x_1^2x_3 - 2x_0x_1x_6 + x_0^2x_1 0$$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